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Montaldi, James (4-WARW); **van Straten, Duco** (D-KSRL)
One-forms on singular curves and the topology of real curve singularities.
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Introduction: "Let $f\colon \mathbf{R}^n, 0 \to \mathbf{R}^{n-1}, 0$ be a real analytic map germ, with $f^{-1}(0)$ a reduced curve germ. In a recent paper [in *Topology and computer science* (Atami, 1986), 347–363, Kinokuniya, Tokyo, 1987], K. Aoki, T. Fukuda and T. Nishimura produced a remarkable algebraic method for computing the number of branches of this curve. Their method is, briefly, to associate to f a map germ $F\colon \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ whose topological degree is equal to the number of branches of $f^{-1}(0)$, and then to use the Eisenbud-Levine theorem to calculate the degree of F as the signature of a quadratic form on the local algebra of F. (We describe it in more detail in §2.)

"The aim of this paper is to generalize the method of Aoki et al. to apply to the case where the curve is not a complete intersection. In the case of a complete intersection, local duality comes into play in the use of the Eisenbud-Levine theorem on F. However, since there is no such map F in the general case, we were led to use local duality and residues on the curve. As usual, the more general setting clarifies the special one.

"Given any meromorphic form α on a curve $\mathcal{C}$, we use the module of Rosenlicht differentials $\omega_{\mathcal{C}}$ of the curve to define two "ramification modules" which measure in some sense the zeros and poles of the form respectively. These modules are finite-dimensional vector spaces, and we prove in §1 that the difference in dimension is preserved under deformation of both the form and the curve. In the case that $\alpha = dg$ for some holomorphic function g, this enables us to find the number of critical points of a small generic deformation of g. Further, we give a simple proof of the fact that the jump in Milnor number in a flat family of curve singularities is equal to the vanishing Euler characteristic.

"In the case that $\mathcal{C}$ and α are real, the 1-form defines an orientation on each connected component of $\mathcal{C} - \{p\}$ (= half-branch), where p is the base point of $\mathcal{C}$. Some of these half-branches will be oriented outwards and some inwards. Moreover, the two ramification modules come with real-valued nondegenerate quadratic forms. We show in §2 that the sum of signatures of these two forms is equal to the difference between the numbers of branches oriented outwards and those oriented inwards. This is related to the classical method of Hermite for calculating the number of real roots of a polynomial as the signature of a quadratic form [see H. M. Weber, *Lehrbuch der Algebra, Bd. I*, Vieweg, Braunschweig, 1895; per bibl.]. We remark that it seems surprising that the two important features of these ramification modules are the difference of the dimensions, but the sum of the signatures."